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String deformations induced by retardation effects in mesons

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Abstract. The rotating-string model is an effective model of mesons, in which the quark and the antiquark are linked by a straight string. We previously developed a method to estimate the retardation effects in this model, but the string was still kept straight. We now go a step further and show that this kind of retardation effects cause a small deviation of the string from the straight line. We first give general arguments constraining the string shape. Then, we find analytical and numerical solutions for the string deformation induced by retardation effects. We finally discuss the influence of the curved string on the energy spectrum of the model.

PACS. 12.39.Ki Relativistic quark model – 14.40.-n Mesons

1 Introduction

The retardation effect between two interacting particles is a relativistic phenomenon, due to the finiteness of the interaction speed. Light mesons are typical systems in which these effects can significantly contribute to the dynamics, since the light quarks can move at a speed close to the speed of light. Inspired by the covariant oscillator quark model [1], we developed in ref. [2] a generalization of the rotating-string model (RSM) [3,4] which must be considered as a first trial to take into account the retardation effects in the mesons. The RSM is an effective model derived from the QCD Lagrangian, describing a meson by a quark and an antiquark linked by a straight string. Both particles are considered as spinless because spin interactions are sufficiently small to be added in perturbation. It has been shown that the RSM was classically equivalent to the relativistic flux tube model [5,6]. This last model, firstly presented in refs. [7,8], yields meson spectra in good agreement with the experimental data [9].

Our method to treat the retardation effects relies on the hypothesis that the relative time between the quark and the antiquark must have a nonzero value. Consequently, in our approach, the evolution parameter of the system is not the common proper time of the quark, the antiquark and the string, but the time coordinate of the center of mass which plays the role of an "average" time.

In a fully covariant theory, the formalism should clearly be independent of the relative time because of the time-reparametrization invariance. Here we start from the usual RSM, in which the retardation effects are neglected (all interactions are instantaneous). The explicit covariance is thus lost and we have to include relative time-dependent terms to "recover" the retardation in the RSM. But, the unphysical relative-time quantum number is removed by imposing constraints coming from covariant approaches [1,10].

We showed in ref. [2] that, in the special case where the quark and the antiquark have the same mass, the part of the total Hamiltonian containing the retardation terms could be treated as a perturbation. This perturbation is a harmonic oscillator in the relative-time variable, with an effective reduced mass and an effective restoring force both depending on eigenstates of the unperturbed Hamiltonian (which is independent of the relative time). The fundamental state of this oscillator —the only relevant one— gives the contribution of the retardation to the masses as well as the relative-time part of the wave function. So, the relative-time wave function is a Gaussian function centered around zero. This point confirms the validity of the usual nonretarded model in first approximation. It is worth mentioning that our retardation term does not destroy the Regge trajectories, *i.e.* the linear relation between the square mass and the spin of the light mesons. Our generalized RSM also allows to reproduce the experimental meson spectrum with a good agreement.

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This previous work is not a fully covariant approach and must be considered as a first step to compute an estimation of the retardation effects in mesons. Some hypothesis were made in order to keep the calculations workable. In particular, the straight-line ansatz was used to describe the string. Although it simplifies the calculations, it is worth noting that the use of a nonvanishing relative time is not really compatible with a straight string. We already gave in ref. [2] a crude estimation of the possible bending of the string due to retardation effects, and our result was compatible with a small deviation from the straight line. However, this point deserves a further study to confirm the validity of our approach. Moreover, these deformations are of intrinsic interest, since they can exist independently of the retardation effects.

Our paper is organized as follows. In sect. 2, we briefly recall the approach developed in ref. [2], the RSM with a nonzero relative time. Then, assuming that the string can be curved, we give arguments constraining its possible shape in sect. 3, and we make a rather general ansatz for the curved string in sect. 4. Using this ansatz, we obtain analytic and numerical solutions for the string shape in sects. 5 and 6. Finally, we sum up our results in sect. 7.

2 Rotating-string model with a nonzero relative time

The RSM with a nonzero relative time has been studied in detail in ref. [2]. So, we simply recall here the main points of this work. Starting from the QCD Lagrangian and neglecting the spin contribution of the quark and the antiquark, the Lagrange function of a meson reads [3] ($\hbar = c = 1$)

$$\mathcal{L} = -m_1\sqrt{\dot{\boldsymbol{x}}_1^2} - m_2\sqrt{\dot{\boldsymbol{x}}_2^2} - a\int_0^1 \mathrm{d}\theta\sqrt{(\dot{\boldsymbol{w}}\boldsymbol{w}')^2 - \dot{\boldsymbol{w}}^2\boldsymbol{w}'^2}\,. \tag{1}$$

The first two terms are the kinetic energy operators of the quark and the antiquark, whose current masses are m_1 and m_2. These two particles are attached by a string with a tension a, described by a well-known Nambu-Goto Lagrangian (see, for example, ref. [11, p. 100]). So, the Nambu-Goto Lagrangian is here used in the framework of an effective QCD theory.

In the following, we will consider only the case $m_1 = m_2 = m$. Introducing auxiliary fields to get rid of the square roots in the Lagrangian (1) and making the straight-line ansatz to describe the string, an effective Lagrangian can be derived [4]

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}\bigg[&2\frac{m^2}{\mu} + a_1\dot{\boldsymbol{R}}^2 + 2a_2\dot{\boldsymbol{R}}\dot{\boldsymbol{r}} - 2c_1\dot{\boldsymbol{R}}\boldsymbol{r}\\ &-2c_2\dot{\boldsymbol{r}}\boldsymbol{r} + a_3\dot{\boldsymbol{r}}^2 + a_4\boldsymbol{r}^2\bigg],\end{aligned} \tag{2}$$

where the coefficients $a_1, a_2, \ldots$, are functions of the auxiliary fields. Their exact expressions can be found in ref. [2]. It is worth mentioning that the auxiliary field μ can be interpreted as the constituent mass of the quark whose current mass is m [6]. $\boldsymbol{r}$ and $\boldsymbol{R}$ are the relative and center-of-mass coordinates, defined by

$$\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \equiv (\sigma, \boldsymbol{r}), \tag{3}$$

$$\boldsymbol{R} = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} \equiv (\bar{t}, \boldsymbol{R}). \tag{4}$$

The straight-line ansatz for the string implies that the string coordinates are given by

$$\boldsymbol{w} = \boldsymbol{R} + \left(\theta - \frac{1}{2}\right)\boldsymbol{r}. \tag{5}$$

Such an ansatz is suggested by lattice QCD calculations, which show that the chromoelectric field between the quark and the antiquark appears to be roughly constant on a straight line joining the two particles [12].

The usual approach is to work with the equal-time ansatz, *i.e.*

$$x_1^0 = x_2^0 = w^0 = \tau = \bar{t}\,. \tag{6}$$

This procedure considerably simplifies the equations, but neglects the relativistic retardation effects. That is why we made in ref. [2] a less restrictive hypothesis: we identified the temporal coordinate of the center of mass with the evolution parameter $\bar{t} = \tau$ and we allowed a nonvanishing relative time σ as in the covariant oscillator quark model [1]. It is then possible to derive from the Lagrangian (2) a set of two equations for the RSM with a nonzero relative time [5,13]

$$\frac{L}{r} = \mu y + \frac{ar}{4y^2}\left(\arcsin y - y\sqrt{1-y^2}\right), \tag{7a}$$

$$H = \frac{p_r^2 + m^2}{\mu} + \mu(1+y^2) + \frac{ar}{y}\arcsin y + \Delta H. \tag{7b}$$

p_r is the radial momentum and y is the transverse velocity of the quark. A further elimination of the auxiliary field μ yields the relativistic flux tube model [5,6].

Equations (7) are identical to those of the usual RSM, but a perturbation of the Hamiltonian, denoted ΔH, is now present. It contains the contribution of the retardation effects and is given by [2]

$$\Delta H = -\frac{\Sigma^2}{2a_3} + \frac{c_2}{a_3}\Sigma\sigma - c_1\sigma - \frac{c_2^2}{2a_3}\sigma^2 + \frac{1}{2}a_4\sigma^2\,, \tag{8}$$

where Σ is the canonical momentum associated with the relative time σ.

Let us now consider the quantized version of our model: $L \to \sqrt{\ell(\ell+1)}$, $[r, p_r] = i$, $[\sigma, \Sigma] = -i$. As the relative time only appears in the perturbation, we can assume that the total wave function reads

$$|\psi(\boldsymbol{r})\rangle = |A(\sigma)\rangle \otimes |R(r)\rangle \otimes |Y_{\ell\,m}(\theta,\phi)\rangle\,, \tag{9}$$

where $|R(r)\rangle$ is a solution of the eigenequation

$$H_0(r)|R(r)\rangle = M_0|R(r)\rangle. \tag{10}$$

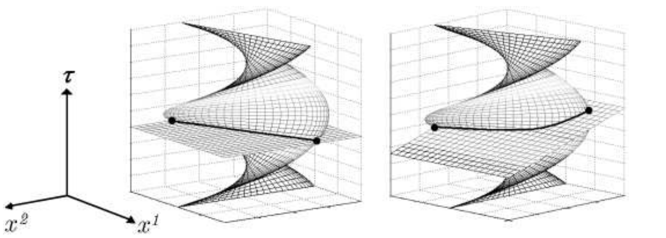

Fig. 1. Left: intersection between a helicoidal string world sheet and a plane $\tau = \tau_0$ ($x^0_q(\tau_0) = x^0_{\bar{q}}(\tau_0)$), the intersection is a straight line. Right: same situation, but now $x^0_q(\tau_0) \neq x^0_{\bar{q}}(\tau_0)$, and the intersection is curved.

$H_0 = H - \Delta H$ is the nonretarded RSM Hamiltonian. Such a problem can be solved for instance by the Lagrange mesh technique [14]. As it is shown in ref. [2], the total mass is written

$$M = M_0 + \Delta M, \tag{11}$$

where the contribution ΔM is given by the fundamental state of the eigenequation

$$\Delta\mathcal{H}(\sigma)|A(\sigma)\rangle = \Delta M|A(\sigma)\rangle, \tag{12}$$

with

$$\Delta\mathcal{H}(\sigma) = -\frac{1}{2\langle a_3\rangle}\left[\Sigma^2 + \langle c_2^2 - a_4 a_3\rangle\sigma^2\right]. \tag{13}$$

In this last formula, the averages are computed with the spatial wave function $|R(r)\rangle$. The requirement that only the fundamental state is relevant eliminates the unphysical degree of freedom introduced with σ, and comes from covariant approaches [1,10].

A detailed study of the retardation term ΔM and of its effect on the meson spectrum can be found in ref. [2]. Let us mention two consequences of eq. (12). Firstly, the retardation contribution ΔM is negative,

$$\Delta M = -\frac{1}{2}\sqrt{\langle c_2^2 - a_4 a_3\rangle/\langle a_3\rangle^2}, \tag{14}$$

and it thus decreases the meson mass. Secondly, the temporal part of the wave function reads

$$A(\sigma) = \left(\frac{\beta}{\pi}\right)^{1/4}\exp\left(-\frac{\beta}{2}\sigma^2\right), \tag{15}$$

with

$$\beta = \sqrt{\langle c_2^2 - a_4 a_3\rangle}. \tag{16}$$

It is a Gaussian function centered around $\sigma = 0$. This provides an interpretation of the equal-time ansatz (6) as the most probable configuration of the system.

It is worth noting that the use of a nonvanishing relative time is not really compatible with the straight-line ansatz. This can be seen by the following simple considerations: Let us assume that the world sheet of the system in the center-of-mass frame is a helicoid area in the case of exactly circular quark orbits. The shape of the string, which we define to be the string coordinates at a constant time $\bar{t}$, is then a straight line for a slice at vanishing relative time and a curve for a slice not at vanishing relative time (see fig. 1). We already gave in ref. [2] a crude estimation of the possible bending of the string due to retardation effects, and our result was compatible with a small deviation from the straight line. The purpose of this paper is to study more carefully the string deformations caused by the nonzero relative time in order to confirm the validity of our approach.

3 Constraints on the string shape

As the coordinates $\boldsymbol{x}_i$ are independent of the parameter θ, the Lagrangian (1) can be rewritten as

$$\mathcal{L} = \int_0^1 \mathrm{d}\theta\tilde{\mathcal{L}}, \tag{17}$$

with

$$\tilde{\mathcal{L}} = -m_1\sqrt{\dot{\boldsymbol{x}}_1^2} - m_2\sqrt{\dot{\boldsymbol{x}}_2^2} - a\sqrt{(\dot{\boldsymbol{w}}\boldsymbol{w}')^2 - \dot{\boldsymbol{w}}^2\boldsymbol{w}'^2}. \tag{18}$$

Several steps are required to obtain a Hamiltonian from the Lagrangian (18). Firstly, one has to find a particular solution, denoted $\boldsymbol{w}^*$, for the string shape. This is in fact the solution of the equations of motion (EOM) of the Nambu-Goto Lagrangian

$$\tilde{\mathcal{L}}_{NG} = -a\sqrt{(\dot{\boldsymbol{w}}\boldsymbol{w}')^2 - \dot{\boldsymbol{w}}^2\boldsymbol{w}'^2}, \tag{19}$$

with the boundary conditions

$$\boldsymbol{w}(1,\tau) = \boldsymbol{x}_1(\tau), \qquad \boldsymbol{w}(0,\tau) = \boldsymbol{x}_2(\tau). \tag{20}$$

Secondly, once $\boldsymbol{w}^*$ is known, it has to be injected in the total Lagrangian (18). Thirdly, one has to compute the momenta defined by

$$p_{i\mu} = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}_i^\mu}. \tag{21}$$

In this picture, we assume that $\boldsymbol{w}^*(\theta,\tau)$ is in fact of the form $\boldsymbol{w}^*(\theta,\boldsymbol{x}_i(\tau))$, because of the boundary conditions (20). Finally, with the momenta (21), the quantity

$$\tilde{\mathcal{H}} = \sum_{i=1}^{2}\boldsymbol{p}_i\dot{\boldsymbol{x}}_i - \tilde{\mathcal{L}} \tag{22}$$

is readily computed, and the total Hamiltonian is thus given by

$$\mathcal{H} = \int_0^1 \mathrm{d}\theta\tilde{\mathcal{H}}. \tag{23}$$

In the usual RSM, one uses the straight-line ansatz (5) together with the equal-time ansatz (6). This provides indeed a correct solution $\boldsymbol{w}^*$ of the EOM [11, p. 122]. The question we want to answer in this section is: Is it possible to find a more general form for the string shape?

Let us suppose that we have such a solution, $\boldsymbol{w}^*$, which thus respects the conditions (20). As we already remarked, $\boldsymbol{w}^*$ depends on the coordinates $\dot{\boldsymbol{x}}_i(\tau)$. So, the momenta (21) read

$$p_{i\mu} = -m_i \frac{\dot{x}_{i\mu}}{\sqrt{\dot{\boldsymbol{x}}_i^2}} - a \frac{N_\rho}{\sqrt{\boldsymbol{N}\dot{\boldsymbol{w}}}} \frac{\partial \dot{w}^\rho}{\partial \dot{x}_i^\mu}, \tag{24}$$

with

$$\boldsymbol{N} = (\dot{\boldsymbol{w}}\boldsymbol{w}')\boldsymbol{w}' - \boldsymbol{w}'^2 \dot{\boldsymbol{w}}. \tag{25}$$

For the sake of simplicity, we have dropped the * on $\boldsymbol{w}$. The Hamiltonian (22), computed with the momenta (24), is expected to vanish since we work in a manifestly covariant formalism. Here, we see that

$$\mathcal{H} = a\left[\sqrt{\boldsymbol{N}\dot{\boldsymbol{w}}} - \sum_{i=1}^{2} \frac{N_\rho}{\sqrt{\boldsymbol{N}\dot{\boldsymbol{w}}}} \frac{\partial \dot{w}^\rho}{\partial \dot{x}_i^\mu} \dot{x}_i^\mu\right]. \tag{26}$$

If we want $\mathcal{H} = 0$, we must satisfy the condition

$$\dot{w}^\rho = \sum_{i=1}^{2} \frac{\partial \dot{w}^\rho}{\partial \dot{x}_i^\mu} \dot{x}_i^\mu. \tag{27}$$

The solution for the constraint (27) is

$$\dot{w}^\rho(\theta,\tau) = \sum_{i=1}^{2} A^\rho_{i\nu}(\theta,\tau)\dot{x}_i^\nu(\tau), \tag{28}$$

where the $A^\rho_{i\nu}(\theta,\tau)$ are unspecified functions. A simple integration and a rewriting of w^ρ in the center of mass and relative coordinates gives

$$\begin{aligned} w^\rho = {} & (A^\rho_{1\nu} + A^\rho_{2\nu})R^\nu + \left[(1-\zeta)A^\rho_{1\nu} - \zeta A^\rho_{2\nu}\right] r^\nu + g^\rho \\ & - \int \mathrm{d}\tilde{\tau} \left\{ (\dot{A}^\rho_{1\nu} + \dot{A}^\rho_{2\nu})R^\nu + \left[(1-\zeta)\dot{A}^\rho_{1\nu} - \zeta \dot{A}^\rho_{2\nu}\right] r^\nu \right\}. \end{aligned} \tag{29}$$

ζ defines the position of the center of mass through the relation $\boldsymbol{R} = \zeta \boldsymbol{x}_1 + (1-\zeta)\boldsymbol{x}_2$. Its value is 1/2 when $m_1 = m_2$. We did not write explicitly the dependences in (θ, τ) to clarify the notations, but g^ρ only depend on θ.

Formula (29) can be simplified by taking the limit $\boldsymbol{r} \to 0$. In this case, we want indeed to obtain $\boldsymbol{w} = \boldsymbol{R}$. This clearly implies that the whole expression (29) can be rewritten by using a single set of functions $A^\rho_\nu \equiv A^\rho_{1\nu}$ as

$$w^\rho = R^\rho + (A^\rho_\nu - \zeta\delta^\rho_\nu) r^\nu - \int \mathrm{d}\tau \dot{A}^\rho_\nu r^\nu. \tag{30}$$

The solution (30) is the general string shape we search for. Solving the corresponding EOM would allow us to determine the functions A^ρ_ν. However, this problem is very complex, and we need to make some simplifications to go further. We want to find a solution which corresponds to a string configuration which does not change in time. So we consider the functions A^ρ_ν to be independent of τ. We also make the assumption that $A^\rho_\nu = A^\rho \delta^\rho_\nu$, without summation on ρ. This implies that we want w^μ depending only on the coordinates R^μ and r^μ. The string has finally the following form:

$$w^\mu(\theta,\tau) = R^\mu(\tau) + \left[A^\mu(\theta) - \zeta\right] r^\mu(\tau), \tag{31}$$

without summation on μ. The remaining unknown quantities are the functions $A^\mu(\theta)$. Let us note that the straight-string ansatz is only a particular case of formula (31), where $A^\mu(\theta) = \theta$.

4 The curved-string ansatz

As we consider here mesons in which the quark and the antiquark have the same mass, only the part of the string linking the center of mass and one quark has to be known. The second part can be computed from the first one by using symmetry arguments, as it will be discussed in sect. 6. Formally, we thus only have to find a solution for $\boldsymbol{w}(\rho,\tau)$ with $\rho \in [0,1]$, such that

$$\boldsymbol{w}(0,\tau) = \boldsymbol{R}, \qquad \boldsymbol{w}(1,\tau) = \boldsymbol{r}_q(\tau), \tag{32}$$

$\boldsymbol{r}_q(\tau)$ being the position of the quark, for instance. ρ is a new parameter covering the half of the string: the center of mass is located at $\rho = 0$ and the quark at $\rho = 1$. As we work in the center-of-mass frame, we can set $\boldsymbol{R} = (\bar{t}, \boldsymbol{0})$. We can then rewrite eq. (31) on the following form:

$$w^0(\rho,\tau) = \bar{t}(\tau) + A^0(\rho) r^0_q(\tau), \tag{33a}$$

$$w^i(\rho,\tau) = A^i(\rho) r^i_q(\tau). \tag{33b}$$

The meson evolves in a plane. We can thus set A^3 equal to zero and use the complex coordinates (w^0, w, w^*) defined by [15]

$$w = \frac{1}{\sqrt{2}}(w^1 + iw^2). \tag{34}$$

The Nambu-Goto Lagrangian is invariant for the reparameterization of the world sheet [11]. It allows us to fix for simplicity

$$\bar{t}(\tau) = \tau \quad \text{and} \quad A^1(\rho) = \rho. \tag{35}$$

In the following, the relations (35) will always be assumed. Let us note that $\bar{t}(\tau) = \tau$ was a hypothesis of the model in ref. [2].

It can be shown that, when $r^0_q(\tau) = 0$,

$$w^0(\rho,\tau) = \tau, \tag{36a}$$

$$w(\rho,\tau) = \frac{r_q(\tau)}{\sqrt{2}} \rho \exp[i\omega\tau], \tag{36b}$$

is a solution of the EOM of the Nambu-Goto, or equivalently, of the Polyakov Lagrangian (see, for example, ref. [15]). It describes a straight string rotating at a constant angular speed. In our coordinates, these EOM read

$$\partial_a \left[\sqrt{-h}\, h^{ab} \partial_b w^\mu\right] = 0, \tag{37a}$$

with h^{ab} the inverse matrix of h_{ab}, given by

$$h_{ab} = -\partial_a w^0 \partial_b w^0 + 2\,\mathrm{Re}\,(\partial_a w^* \partial_b w)\,. \tag{37b}$$

The indices (a, b) label the parameters (τ, ρ), and h_{ab} is the induced metric on the string world sheet.

In our previous work on the retardation effects, we used the following ansatz [2]:

$$w^0(\rho,\tau) = \tau + \rho\,\sigma_q(\tau), \tag{38a}$$

$$w(\rho,\tau) = \frac{r_q(\tau)}{\sqrt{2}}\,\rho\exp(i\omega\tau). \tag{38b}$$

However, the string defined by eqs. (38) is not a solution of the EOM. Indeed, as $\sigma_q(\tau)$ is assumed to be small (we showed that the retardation can be treated as a perturbation), we can write the EOM at the first order in σ_q. After some calculations, one can check from relations (37) that σ_q must satisfy the following equation:

$$r_q^2\rho^2\ddot{\sigma}_q - 2r_q\dot{r}_q\rho^2\dot{\sigma}_q + \left[\left(r_q^2\omega^2 - r_q\ddot{r}_q + 2\dot{r}_q^2\right)\rho^2 - 2\right]\sigma_q = 0. \tag{39}$$

As σ_q is only a function of τ, eq. (39) must be satisfied for every value of ρ. In particular, when $\rho = 0$, we observe that $\sigma_q = 0$ is the only possible solution. This confirms that a straight string is not compatible with a nonzero relative time.

Finally, it appears quite natural that the string could be curved because of the addition of two effects: the rotation of the meson and the finiteness of the gluon speed. A curved string can be described by the ansatz

$$w^0(\rho,\tau) = \tau + \rho\,\sigma_q(\tau), \tag{40a}$$

$$w(\rho,\tau) = \frac{r_q(\tau)}{\sqrt{2}}\,[\rho + if(\rho)]\exp[i(\omega\tau + \phi(\tau))], \tag{40b}$$

where the spatial deformation $f(\rho)$ has been introduced as a counterpart to the relative time $\sigma_q(\tau)$. Moreover, the eventuality of a nonconstant rotation speed is taken into account through the angular acceleration $\phi(\tau)$. If $\sigma_q = 0$, our ansatz reduces to the one of ref. [15], where the string deformations due to angular acceleration are studied. Equations (40) clearly describe a curved string, as it can be seen by rewriting it when $\tau = 0$ with $\phi(0) = 0$. Then, we have simply $w^1 \propto \rho$ and $w^2 \propto f(\rho)$.

5 Analytic solution

Finding an exact expression of $f(\rho)$ which satisfies the EOM of the Nambu-Goto Lagrangian is a very complex problem, out of the scope of this paper. The use of approximations appears necessary in order to deal with workable equations. As already pointed out in ref. [2], some arguments show that the deformation of the string should be small. Assuming that point, we will linearize the EOM in $f(\rho)$ and its derivatives. The angular acceleration will also be considered as small, as it is done in ref. [15]. Following the hypothesis of this last reference, we will only keep the linear terms in ϕ and its derivatives. Terms like ϕf, $\phi\sigma_q$, $\phi\dot{\sigma}_q$, $\phi\dot{r}_q, \ldots$ will thus also be neglected. Moreover, we consider that $\ddot{r}_q$, $\ddot{\sigma}_q \approx 0$. Consequently, our solution will only be valid in the case of small radial excitations.

After a tedious algebra, we find that the EOM (37) are satisfied by the curved string (40) if

$$\begin{aligned}&\left(1 - A_1\rho - A_2\rho^2 + A_3\rho^3\right)\partial_\rho^2 f\\ &+(B_1 - B_2\rho)\,(\rho\partial_\rho f - f) = C + D\rho - E\rho^2\,,\end{aligned} \tag{41}$$

with

$$A_1 = \sigma_q\frac{\dot{r}_q}{r_q} - 3\dot{\sigma}_q\,, \tag{42a}$$

$$A_2 = \omega^2 r_q^2 + \dot{r}_q^2 + 2\sigma_q\dot{\sigma}_q\frac{\dot{r}_q}{r_q} - 3\dot{\sigma}_q{}^2\,, \tag{42b}$$

$$\begin{aligned}A_3 = {}&\omega^2\sigma_q r_q\dot{r}_q + \sigma_q\frac{\dot{r}_q^3}{r_q} - \omega^2\dot{\sigma}_q r_q^2 - \dot{\sigma}_q\dot{r}_q^2\\ &-\sigma_q\dot{\sigma}_q{}^2\frac{\dot{r}_q}{r_q} + \dot{\sigma}_q{}^3\,,\end{aligned} \tag{42c}$$

$$B_1 = \omega^2 r_q^2 + 3\omega^2\sigma_q^2 + 2\sigma_q^2\frac{\dot{r}_q^2}{r_q^2} - 2\sigma_q\dot{\sigma}_q\frac{\dot{r}_q}{r_q}\,, \tag{42d}$$

$$\begin{aligned}B_2 = {}&2\omega^2\sigma_q r_q\dot{r}_q + 2\sigma_q\frac{\dot{r}_q^3}{r_q} + \omega^2\dot{\sigma}_q r_q^2 - 3\omega^2\dot{\sigma}_q\sigma_q^2 - 2\dot{\sigma}_q\dot{r}_q^2\\ &+2\sigma_q^2\dot{\sigma}_q\frac{\dot{r}_q^2}{r_q^2} + 2\sigma_q\dot{\sigma}_q{}^2\frac{\dot{r}_q}{r_q}\,,\end{aligned} \tag{42e}$$

$$C = 2\sigma_q\omega, \tag{42f}$$

$$D = 2\omega\sigma_q\dot{\sigma}_q - 2\omega\sigma_q^2\frac{\dot{r}_q}{r_q} + r_q^2\ddot{\phi}, \tag{42g}$$

$$\begin{aligned}E = {}&-\omega^3\sigma_q r_q^2 + \omega^3\sigma_q^3 - 2\omega\sigma_q\dot{r}_q^2 + 2\omega\sigma_q^3\frac{\dot{r}_q^2}{r_q} + 2\omega\dot{\sigma}_q r_q\dot{r}_q\\ &-2\omega\sigma_q^2\dot{\sigma}_q\frac{\dot{r}_q}{r_q}\,,\end{aligned} \tag{42h}$$

and the boundary conditions

$$f(0) = f(1) = 0. \tag{43}$$

Conditions (43) are in fact equivalent to the initial boundary conditions (32).

Before performing a numerical resolution of the differential equation (41), we can find an approximate analytic solution by developing $f(\rho)$ in powers of ρ,

$$f(\rho) = \sum_{n=0}^{\infty} a_n\rho^n\,. \tag{44}$$

Keeping in this series only the terms which satisfy eqs. (41) and (43) at the second order in ρ, one can find that

$$f(\rho) = f_C(\rho) + f_D(\rho) + f_E(\rho), \tag{45}$$

with

$$f_C(\rho) = -\frac{C}{2}\,\rho\,(1-\rho), \tag{46a}$$

$$f_D(\rho) = -\frac{(D+CA_1)}{6}\,\rho\,(1-\rho^2), \tag{46b}$$

$$f_E(\rho) = \frac{1}{12}\,[E - A_1(D+CA_1) - C(A_2 - B_1/2)] \times\rho\,(1-\rho^3). \tag{46c}$$

The solution at the lowest order is $f_C(\rho)$. It is a parabola whose maximum is reached in $\rho = 1/2$. The next functions, $f_D(\rho)$ and $f_E(\rho)$, shift slightly the maximum at a value $\rho < 1/2$. A graphical representation of the solution (45) is given in the next section. It is worth mentioning that in the case of a vanishing angular momentum, $\omega = 0$, the solution is trivially $f(\rho) = 0$. Even when the retardation is included, the string is straight when the angular momentum is zero.

We can check that if $\sigma_q = 0$, our solution reduces to

$$f(\rho)|_{\sigma_q=0} = -\frac{r_q^2\ddot{\phi}}{6}\rho(1-\rho^2), \tag{47}$$

which is precisely the result of ref. [15]. However, as we are only interested in the retardation effects, we will take $\ddot{\phi} = 0$ in the following.

6 Numerical solution

6.1 Evaluation of the coefficients

The previous section gave us a qualitative idea of the string shape. But, to complete our analysis, we need an estimation of the magnitude of the deformation. To do this, we have to compute numerically the different coefficients (42), not in the classical framework we used up to now, but in a quantized model. The coefficients (42) should then be seen as operators whose average value has to be computed. What should be done rigorously is to rewrite our RSM Hamiltonian with a curved string solution of eq. (41) and compute the eigenstates of this model to average the operators (42), in an analog way of what is done in ref. [16]. But here, we only want to have a first estimation of the string deformation. That is why, as we considered that the deformation was small, it appears reasonable to compute the mean values of the coefficients (42) with the states of our particular RSM introduced in ref. [2]. Let us remark that, if we want to be consistent with the notations for r and σ in sect. 2, we have to make the following substitutions:

$$r_q \to r/2, \qquad \sigma_q \to \sigma/2, \tag{48}$$

the angular speed ω being not modified. Our procedure will be to replace each positive definite operator X appearing in the definitions (42) by $\sqrt{\langle X^2\rangle}$, a quantity easy to compute with our numerical method [14]. This simple computation, neglecting symmetrization problems, will only give us a rough estimation of the different coefficients, but it is sufficient for our purpose.

We already mentioned that our numerical method, the Lagrange mesh method, ensures us to know the radial wave function $|R(r)\rangle$. The average values $\langle \boldsymbol{p}^2\rangle$ and $\langle r^2\rangle$ are easy to compute [14]. Moreover, it is shown in ref. [6] that the angular speed can be approximated by

$$\omega \approx \frac{\sqrt{\ell(\ell+1)}}{\langle r^2\rangle\left(\left\langle\sqrt{\boldsymbol{p}^2+m^2}\right\rangle + a\langle r\rangle/6\right)}\,, \tag{49}$$

where ℓ is the orbital angular momentum. The last spatial term we need is $\langle \dot{r}^2\rangle$, which is the radial part of $\langle \dot{\boldsymbol{r}}^2\rangle$. It can be computed, from the Lagrangian (2), that [6]

$$\left\langle p_r^2\right\rangle = \left\langle (a_3\dot{r} - c_2 r)^2\right\rangle. \tag{50}$$

Using the fact that $\langle c_2\rangle \approx 0$ [2], we have

$$\left\langle \dot{r}^2\right\rangle \approx \frac{1}{\langle a_3\rangle^2}\left[\left\langle p_r^2\right\rangle - \left\langle c_2^2\right\rangle\left\langle r^2\right\rangle\right]. \tag{51}$$

We turn now our attention to the terms involving the relative time. With the temporal wave function given by eq. (15), it is easily computed that

$$\sqrt{\langle \sigma^2\rangle} = \frac{1}{\sqrt{2\beta}}\,. \tag{52}$$

$\langle \dot{\sigma}^2\rangle$ can be computed in analogy with $\langle \dot{r}^2\rangle$. From the Lagrangian (2), it can be shown that [2]

$$\left\langle \Sigma^2\right\rangle = \left\langle (c_2\sigma - a_3\dot{\sigma})^2\right\rangle. \tag{53}$$

Since

$$\left\langle \Sigma^2\right\rangle = \frac{\beta}{2}\,, \tag{54}$$

we have

$$\left\langle \dot{\sigma}^2\right\rangle \approx \frac{\beta}{2\langle a_3\rangle^2}\left[1 - \frac{\left\langle c_2^2\right\rangle}{\beta^2}\right]. \tag{55}$$

We are now able to compute, at least in first approximation, the coefficients (42).

6.2 Results

The more interesting case is the meson formed of two massless quarks. The deformation of the string is indeed expected to be maximal in this case since the relativistic effects are the most important. Of course, we will choose $\ell \neq 0$ to observe a nonzero deformation. With $m = 0$ and the standard value $a = 0.2\,\mathrm{GeV}^2$, we can compute the needed average values thanks to formulas (49) to (55). They are given in table 1 for three different states.

A numerical integration of eq. (41) to obtain the solution $f(\rho)$ with the boundary conditions (43) can be performed by using the values of table 1. As mentioned in sect. 4, one half of the string is simply given by the couples

Table 1. Average values of quantities involved in the coefficients (42), for different states. For simplicity, the quantities of the form $\sqrt{\langle X^2 \rangle}$ appearing in the first column are denoted as X. We recall that $r_q = r/2$ and $\sigma_q = \sigma/2$.

$(n+1)L$	$1P$	$1F$	$2P$
r_q (GeV^{-1})	2.526	3.376	3.411
σ_q (GeV^{-1})	1.093	1.072	1.122
ω (GeV)	0.086	0.087	0.036
$\dot{r}_q$	0.459	0.303	0.592
$\dot{\sigma}_q$	0.342	0.257	0.252

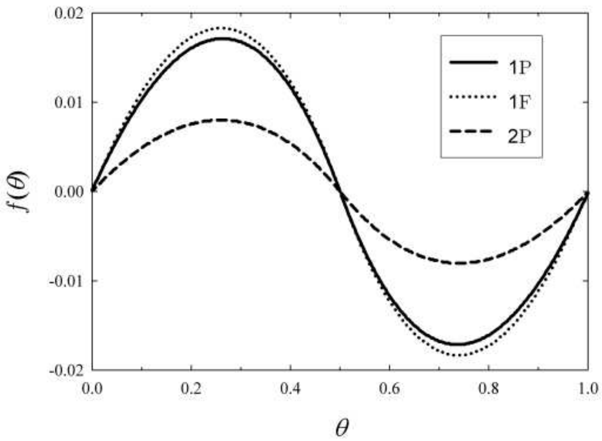

Fig. 2. Numerical solution of eq. (41), giving the string shape between the quark and the antiquark for the states considered in table 1. $f(\theta) = 0$ corresponds to the straight-line ansatz.

$\{\rho, f(\rho)\}$. Once this solution is known, we can compute every couple $\{\theta, f(\theta)\}$ between the quark and the antiquark by a central symmetry with respect to the center of mass. The numerical solution of eq. (41) is plotted in fig. 2 for the three states considered in table 1.

We found that the deformation is maximal for the $1F$ state, and then decreases when the quantum numbers increase. This is a consequence of our approach, since we noticed in ref. [2] that the contribution of the retardation effects decreases when the dynamical quark mass $\langle \sqrt{\boldsymbol{p}^2 + m^2} \rangle$ increases. In every case, the deformation $f(\theta)$ is small with respect to one. This is an *a posteriori* validation of our choice to consider small deformations only.

In fig. 3, we compare the numerical solution for the $1P$ state, the lowest state in which the deformation occurs, with three analytic approximations of this solution, given by relations (45) and (46). It appears that f_C is an upper bound of the deformation, and that $f_C + f_D$ is sufficient to correctly approximate the numerical solution.

Since the string brings an energetic contribution to the meson which is proportional to its length, it is interesting to evaluate the ratio between the lengths of both

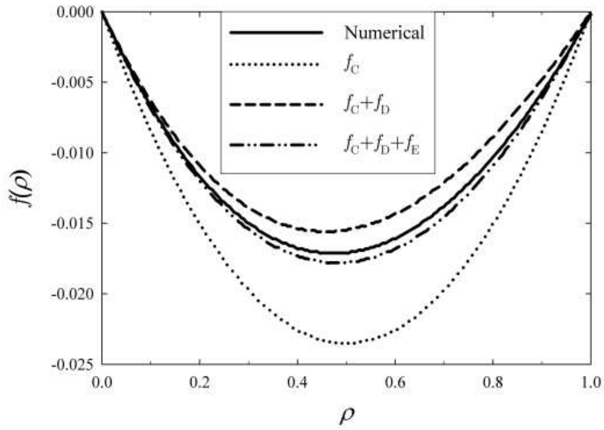

Fig. 3. Comparison between the numerical solution of eq. (41) for the $1P$ state and the analytical formula (45).

the curved and the straight strings. It is given by

$$\frac{\Delta L}{L} = \left[\int_0^1 \mathrm{d}\rho \sqrt{1 + (\partial_\rho f)^2} \right] - 1, \tag{56}$$

$$\approx \int_0^1 \mathrm{d}\rho \frac{(\partial_\rho f)^2}{2}. \tag{57}$$

For a small deformation, as it is the case here, we can obtain an upper bound for $\Delta L/L$ by using f_C instead of f. To the second order in C, we find that

$$\frac{\Delta L}{L} \leq \frac{C^2}{24}, \tag{58}$$

with $C = 2\sigma_q \omega$. It is maximal in the $1F$ state. We finally obtain the upper bound

$$\frac{\Delta L}{L} \leq 2 \times 10^{-3}. \tag{59}$$

The length of the string is only modified by some tenths of percent, because of the bending induced by retardation effects. As the typical mass scale for the mesons is the 1–2 GeV, the correction due to curved string is around 1–4 MeV, as it is observed in ref. [16]. The contribution of the bending of the string to the mass spectrum seems thus very small. This is also small compared with the retardation contribution, which can be around 100 MeV for massless quarks.

7 Conclusion

We developed in this paper the idea that, if the retardation effects are included in the rotating-string model, the string linking the quark and the antiquark cannot remain a straight line. We found a relation constraining its shape, and obtained a general form for the string. The straight-line ansatz, which is valid in the equal-time approximation, appears to be only a particular case of our general form. When the retardation is included with the method proposed in ref. [2], we showed that a bending of the string must be *a priori* taken into account, and we proposed a general ansatz defining a curved string. With this ansatz, and assuming *a priori* that the deformation of the string is small, we derived a differential equation giving its shape. Analytical and numerical solutions of this equation can be found. Roughly, the deformed string has a parabolic shape, with a small amplitude. The amplitude of the deformation decreases when the quantum numbers increase; but it is zero for a vanishing angular momentum. We finally argued that the contribution of the bending of the string to the mass spectrum is around 1–4 MeV. The typical meson mass scale is around 1–2 GeV, and the retardation contribution computed in a previous work with a straight string is always smaller than 100 MeV. So, the effects of the curvature are very weak, although rather interesting from a theoretical point of view. However, if one is mainly interested in the computation of a mass spectrum, we can conclude from our present work that the rotating string with nonzero relative time can give satisfactory results, event with a straight-string approximation.

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